

Microscopic model versus systematic low-energy effective field theory for a doped quantum ferromagnet

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We consider a microscopic model for a doped quantum ferromagnet as a test case for the systematic low-energy effective field theory for magnons and holes, which is constructed in complete analogy to the case of quantum antiferromagnets. In contrast to antiferromagnets, for which the effective field theory approach can be tested only numerically, in the ferromagnetic case, both the microscopic and the effective theory can be solved analytically. In this way, the low-energy parameters of the effective theory are determined exactly by matching to the underlying microscopic model. The low-energy behavior at half-filling as well as in the single- and two-hole sectors is described exactly by the systematic low-energy effective field theory. In particular, for weakly bound two-hole states the effective field theory even works beyond perturbation theory. This lends strong support to the quantitative success of the systematic low-energy effective field theory method not only in the ferromagnetic but also in the physically most interesting antiferromagnetic case.

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I. INTRODUCTION

Achieving a quantitative understanding of the doped antiferromagnetic precursors of high-temperature superconductors is a great challenge in condensed matter physics. In particular, away from half-filling, Monte Carlo simulations of these strongly correlated electron systems suffer from a very severe sign problem. Also analytic calculations in underlying microscopic Hubbard or t - J -type models are not fully systematic but suffer from uncontrolled approximations. Particle physicists face similar challenges in the physics of the strong interactions between quarks and gluons. Remarkably, the low-energy physics of pions—the pseudo-Goldstone bosons of the spontaneously broken $SU(2)_L \times SU(2)_R$ chiral symmetry of QCD—is described quantitatively by a systematic effective field theory,^{1–4} known as chiral perturbation theory. Similarly, the low-energy physics of the spin waves or magnons—the Goldstone bosons of the spontaneously broken $SU(2)_s$ spin symmetry in an antiferromagnet—is also captured by a systematic effective field theory.^{5–11} Early attempts to include doped holes into the effective description of antiferromagnets are described in Refs. 12–15. Motivated by the quantitative success of baryon chiral perturbation theory^{16–19} for pions and nucleons in QCD, fully systematic low-energy effective field theories have been developed for hole-doped antiferromagnets both on a square^{20,21} and on a honeycomb lattice,²² as well as for electron-doped antiferromagnets on a square lattice.²³ The resulting systematic effective field theories have been used to study magnon-mediated two-hole^{21,24} and two-electron bound states²³ as well as spiral phases in the staggered magnetization order parameter.^{22,23,25} The quantitative correctness of the magnon effective field theory has been demonstrated in great detail at permille level accuracy by comparison with Monte Carlo simulations of the quantum Heisenberg model using the very

efficient loop-cluster algorithm.^{26–28} Similarly, the single-hole sector of the t - J model has been simulated both on the square^{29,30} and on the honeycomb lattice.³¹ Indeed, the observed location of the hole pockets in the Brillouin zone has provided important input for the construction of the various systematic effective field theories for doped antiferromagnets.

In general, low-energy effective field theories cannot be derived rigorously from the underlying microscopic physics. Instead, one performs a detailed symmetry analysis of the underlying theory and constructs all terms in the effective Lagrangian that are invariant, order by order in a systematic derivative expansion. Each term is then endowed with an *a priori* undetermined low-energy parameter. In particular, the values of these parameters are not fixed by symmetry considerations, but must be determined by matching to the underlying microscopic system. This can be done by comparison with either experiment or numerical simulations. Only in exceptional cases the underlying microscopic model can be solved analytically and the low-energy parameters can be determined exactly. One such case is the ferromagnetic Heisenberg model whose low-energy physics was analytically derived by Dyson.³² The corresponding low-energy effective theory was constructed by Leutwyler³³ and discussed in great detail in Refs. 34–36. Remarkably, in contrast to the effective theory for antiferromagnets, the effective theory for ferromagnets contains an additional Wess-Zumino term whose quantized prefactor is the total magnetization. The values of the magnetization and of the spin stiffness—the other leading order low-energy parameter of a ferromagnet—can be easily read off from Dyson’s analytic solution of the underlying microscopic Heisenberg model. Thanks to the analytic solvability of the ferromagnetic Heisenberg model, in this case the predictions of the effective theory can be verified rigorously. Indeed, once the low-energy parameters

have been fixed by matching to the underlying system, in the low-energy domain the effective theory yields exactly the same results as the Heisenberg model. It is interesting to note that the calculations in the effective theory are much simpler than those in the microscopic model.

The effective field theories for doped antiferromagnets mentioned above have again been constructed based on symmetry considerations. However, in that case the underlying two-dimensional Hubbard or t - J -type models cannot be solved analytically, and one must hence rely on numerical methods for fixing the low-energy parameters and for verifying the validity of the low-energy effective theory. In this paper, we consider a microscopic Hubbard-type model for a doped ferromagnet which can be solved analytically. Furthermore, the corresponding low-energy effective theory can be constructed in exactly the same way as in the antiferromagnetic case. By showing explicitly that the microscopic and the effective theory of the doped ferromagnet yield identical results, we lend further support to the general construction principle for the effective theories. For simplicity, our analytic study will be performed in one spatial dimension, but the extension to higher dimensions is straightforward. It should be noted that in one spatial dimension the antiferromagnetic Heisenberg model is analytically solvable by the Bethe ansatz.³⁷ According to Haldane's conjecture,³⁸ the corresponding low-energy effective theory is a two-dimensional $O(3)$ nonlinear σ model at vacuum angle $\theta = \pi$. As first noted by Lieb and Wu, in one dimension even the Hubbard model can be solved analytically.^{39,40} In particular, these authors have shown that this model has no Mott transition. We prefer to consider the ferromagnetic model because it is easier to solve analytically and because its low-energy effective theory is similar to the one of the doped antiferromagnets. It should be pointed out that our ferromagnetic model is not meant to provide a realistic description of ferromagnetism in actual materials. This would require two bands as well as Hund rule couplings.⁴¹ A more realistic model for doped quantum ferromagnets has been discussed and solved analytically in Ref. 42 and the related polaron problems have been discussed in Ref. 43. For simplicity, here, we impose ferromagnetism by including a corresponding coupling by hand. Still, the range of applicability of the effective theory to be constructed in this paper goes beyond our ferromagnetic model, as the effective theory applies to any system exhibiting the same symmetries and symmetry breaking pattern as the microscopic model considered here.

While the calculations in the microscopic model are non-perturbative, some of the corresponding calculations in the effective theory are based on perturbation theory. Indeed, it is a big advantage of the effective theory approach to Goldstone boson physics that perturbation theory provides quantitatively correct results in a systematic low-energy expansion. While Goldstone bosons are derivatively and thus weakly coupled at low energies, the contact interactions between two holes may very well be strong, thus requiring a nonperturbative treatment not only of the microscopic model, but also of the effective field theory. A similar situation arises in the effective field theory approach to the strong interactions between nucleons and pions—the Goldstone bosons of the spontaneously broken chiral symmetry of

QCD. Systematic low-energy effective field theories have also been used in studying light nuclei.^{44–53} Then, the short-range repulsion between two nucleons is strong, which implies that the effective theory must be treated nonperturbatively. In that case, it is still an unsettled theoretical question how this can be achieved fully systematically. In particular, there are various power-counting schemes, due to Weinberg,⁴⁴ as well as due to Kaplan *et al.*,⁴⁵ which are both not fully satisfactory. Recently, an interesting modification of the Kaplan-Savage-Wise scheme has been proposed,⁵³ and it remains to be seen whether this will finally resolve this issue. In contrast to QCD or doped antiferromagnets, the ferromagnetic model studied here has the advantage that it can be solved analytically. Hence, one may reach a deeper understanding of the subtle nonperturbative fermion dynamics. For this purpose, we will investigate the two-hole sector in the effective field theory and will then compare with the analytic results of the underlying microscopic model. Hence, the doped ferromagnet is a system in which systematic approaches to nonperturbative problems in effective field theory can be tested. Thus, the investigations in this or related models may also have an impact on the corresponding issues arising in the context of the strong interactions.

The paper is organized as follows. In Sec. II, the underlying microscopic model is presented and its symmetry properties are investigated in detail. The model is then solved at half-filling, as well as in the one- and two-hole sectors. In particular, the dispersion relations of magnons and holes, as well as the binding energy of two holes and the two-hole scattering states are determined analytically. In Sec. III, the corresponding low-energy effective field theory is constructed using the nonlinear realization of the spontaneously broken $SU(2)_s$ spin symmetry. In particular, the hole fields are included in the same way as for a doped antiferromagnet. In Sec. IV, magnons, single holes, as well as two-hole scattering and two-hole bound states are investigated in the effective field theory framework. The *a priori* undetermined low-energy parameters are fixed by matching to the underlying microscopic system, and it is verified explicitly that the predictions of the effective theory agree exactly with those of the microscopic model. Finally, Sec. V contains our conclusions. Some technical details are presented in the Appendix.

II. CONSTRUCTION AND SOLUTION OF A MICROSCOPIC MODEL FOR A DOPED FERROMAGNET

In this section, we construct a Hubbard-type microscopic model for a doped ferromagnet, investigate its symmetries, and then solve it in the zero-, one-, and two-hole sectors.

A. Microscopic model for ferromagnetism

Let us construct a microscopic model describing the hopping of fermions on a one-dimensional lattice with spacing a , with the Hamiltonian

$$H = -t \sum_x (c_x^\dagger c_{x+a} + c_{x+a}^\dagger c_x) - J \sum_x \vec{S}_x \cdot \vec{S}_{x+a} + \frac{U}{2} \sum_x (c_x^\dagger c_x - 1)^2. \quad (2.1)$$

The creation and annihilation operators for fermions at a site $x = an$, $n \in \mathbb{Z}$, with spin $s = \uparrow, \downarrow$, are given by

$$c_x^\dagger = (c_{x\uparrow}^\dagger, c_{x\downarrow}^\dagger), \quad c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix}. \quad (2.2)$$

They obey the standard anticommutation relations

$$\{c_{xs}^\dagger, c_{x's'}\} = \delta_{xx'} \delta_{ss'}, \quad \{c_{xs}, c_{x's'}\} = \{c_{xs}^\dagger, c_{x's'}^\dagger\} = 0. \quad (2.3)$$

Putting $\hbar=1$, the spin operator at the site x is given by

$$\vec{S}_x = c_x^\dagger \frac{\vec{\sigma}}{2} c_x, \quad (2.4)$$

where $\vec{\sigma}$ denotes the Pauli matrices. Let us discuss the various terms in the Hamiltonian above. The term proportional to t describes hopping of fermions by one lattice spacing, i.e., it represents the kinetic energy. The parameter $J>0$ is a ferromagnetic exchange coupling constant, while the term proportional to $U>0$ describes an on-site Coulomb repulsion. As mentioned earlier, this Hamiltonian does not provide a realistic description of real ferromagnetic materials. We consider it because it is analytically solvable at low energies and can thus be used to test the corresponding effective theory.

The microscopic model defined by Eq. (2.1) has various symmetries, which we are going to discuss now. It is straightforward to confirm that the Hamiltonian commutes with the total spin

$$[H, \vec{S}] = 0, \quad \vec{S} = \sum_x \vec{S}_x, \quad (2.5)$$

and is thus invariant under global $SU(2)_s$ spin rotations. As we will see later, the $SU(2)_s$ symmetry is spontaneously broken down to the subgroup $U(1)_s$ by the formation of a uniform magnetization. It should be noted that this is not in contradiction with the Mermin-Wagner theorem. The generators of another symmetry—a non-Abelian $SU(2)_Q$ extension of the Abelian $U(1)_Q$ fermion number^{54,55}—are given by

$$Q^+ = \sum_x (-1)^{x/a} c_{x\uparrow}^\dagger c_{x\downarrow}^\dagger, \quad Q^- = \sum_x (-1)^{x/a} c_{x\downarrow} c_{x\uparrow},$$

$$Q^3 = \sum_x \frac{1}{2} (c_x^\dagger c_x - 1). \quad (2.6)$$

The factor $(-1)^{x/a}$ distinguishes between the two sublattices A and B of even and odd sites. Unlike for an antiferromagnet, it may seem unnatural to make such a distinction for a ferromagnet. However, as we will see later on, the introduction of two sublattices is also important for a ferromagnet, as it will allow us to correctly identify the transformation properties of holes and electrons in the effective theory. It is straightforward to convince oneself that the Hamiltonian is indeed invariant, i.e.,

$$[H, \vec{Q}] = 0, \quad \vec{Q} = (Q^1, Q^2, Q^3), \quad Q^\pm = Q^1 \pm iQ^2. \quad (2.7)$$

It should be pointed out that the $SU(2)_Q$ symmetry would be explicitly broken down to $U(1)_Q$ if hopping terms between sites belonging to the same sublattice would be included in the Hamiltonian. Furthermore, it is worth noting that the generators of $SU(2)_s$ commute with those of $SU(2)_Q$.

A displacement D by one lattice spacing is generated by the unitary operator D , which acts as

$${}^D c_x = D^\dagger c_x D = c_{x+a}. \quad (2.8)$$

By relabeling the sum over the lattice points, it is easy to show that $[H, D]=0$. Another discrete symmetry is the spatial reflection R , which acts as

$${}^R c_x = R^\dagger c_x R = c_{-x}. \quad (2.9)$$

Again, by relabeling the sum over the lattice points, it follows that $[H, R]=0$. Another important symmetry is time reversal which is implemented by an antiunitary operator T .

It is useful to introduce a matrix-valued fermion operator

$$C_x^A = \begin{pmatrix} c_{x\uparrow} & c_{x\downarrow}^\dagger \\ c_{x\downarrow} & -c_{x\uparrow}^\dagger \end{pmatrix}, \quad x \in A,$$

$$C_x^B = \begin{pmatrix} c_{x\uparrow} & -c_{x\downarrow}^\dagger \\ c_{x\downarrow} & c_{x\uparrow}^\dagger \end{pmatrix}, \quad x \in B. \quad (2.10)$$

Under combined transformations $g \in SU(2)_s$ and $\Omega \in SU(2)_Q$, it transforms as

$$\bar{Q} C_x' = g C_x \Omega^T. \quad (2.11)$$

Under the displacement symmetry, one obtains

$${}^D C_x^A = C_{x+a}^B \sigma_3, \quad {}^D C_x^B = C_{x+a}^A \sigma_3. \quad (2.12)$$

The appearance of the Pauli matrix σ_3 is due to the factor $(-1)^{x/a}$. Under the spatial reflection R , which turns x into $Rx=-x$, one obtains

$${}^R C_x = C_{-x}. \quad (2.13)$$

The Hamiltonian can now be expressed in a manifestly $SU(2)_{s^-}$, $SU(2)_{Q^-}$, D -, and R -invariant form

$$H = -\frac{t}{2} \sum_x \text{Tr}[C_x^\dagger C_{x+a} + C_{x+a}^\dagger C_x]$$

$$- \frac{J}{16} \sum_x \text{Tr}[C_x^\dagger \vec{\sigma} C_x] \cdot \text{Tr}[C_{x+a}^\dagger \vec{\sigma} C_{x+a}]$$

$$+ \frac{U}{12} \sum_x \text{Tr}[C_x^\dagger C_x C_x^\dagger C_x]. \quad (2.14)$$

B. Eigenstates for electrons and holes

We will now construct electron and hole states above a half-filled ground state containing up-spin fermions at each lattice site. The corresponding vacuum state is given by

$$|v\rangle = \prod_x c_{x\uparrow}^\dagger |0\rangle, \quad (2.15)$$

where $|0\rangle$ represents an empty lattice without any fermions. Indeed, acting with the Hamiltonian, one obtains

$$H|v\rangle = E_v|v\rangle, \quad E_v = -\frac{J}{4}N, \quad (2.16)$$

i.e., $|v\rangle$ is indeed an eigenstate, with the vacuum energy E_v proportional to the number of lattice sites N . The total spin of the state $|v\rangle$ is $S=N/2$. By acting with the lowering operator $S^- = \sum_x S_x^-$ on the vacuum state $|v\rangle$, one can construct the other ground states belonging to the same $SU(2)_s$ multiplet, which contains $2S+1=N+1$ degenerate states.

Let us now construct a somewhat unconventionally normalized hole state of momentum p

$$|hp\rangle = \sum_x \exp(ipx) c_{x\uparrow} |v\rangle = c_{p\uparrow} |v\rangle. \quad (2.17)$$

In order to check whether this is an eigenstate, we compute

$$H|hp\rangle = ([H, c_{p\uparrow}] + c_{p\uparrow} H)|v\rangle = (E_h(p) + E_v)|hp\rangle, \quad (2.18)$$

which shows that $|hp\rangle$ is indeed an energy eigenstate. One obtains the energy-momentum dispersion relation of a hole as

$$E_h(p) = \frac{J}{2} + \frac{U}{2} + 2t \cos(pa) = \frac{J}{2} + \frac{U}{2} + 2t - ta^2 \hat{p}^2, \quad (2.19)$$

where we have introduced $\hat{p} = \frac{2}{a} \sin(pa/2)$. This periodic function has minima at $p = (2n-1)\pi/a$, with $n \in \mathbb{Z}$. Expanding around $p = \pi/a$, we obtain

$$E_h(p) = \frac{J}{2} + \frac{U}{2} - 2t + ta^2 \left(p - \frac{\pi}{a} \right)^2 + \mathcal{O} \left(\left(p - \frac{\pi}{a} \right)^4 \right). \quad (2.20)$$

The holes are massive objects and their dispersion relation is given by

$$E_h(p) = M_h + \frac{(p - \pi/a)^2}{2M'_h} + \mathcal{O} \left(\left(p - \frac{\pi}{a} \right)^4 \right), \quad (2.21)$$

with the rest mass M_h and the kinetic mass M'_h given by

$$M_h = \frac{J}{2} + \frac{U}{2} - 2t, \quad M'_h = \frac{1}{2ta^2}. \quad (2.22)$$

Since the theory is nonrelativistic, the rest mass M_h and the kinetic mass M'_h need not to be the same.

Similarly, we construct electron states

$$|ep\rangle = \sum_x \exp(-ipx) c_{x\downarrow}^\dagger |v\rangle = c_{p\downarrow}^\dagger |v\rangle, \quad (2.23)$$

and we compute

$$Q^- |ep\rangle = - \left| h - \left(p + \frac{\pi}{a} \right) \right\rangle. \quad (2.24)$$

The $SU(2)_Q$ symmetry then implies that the energy of an electron is given by (Fig. 1)

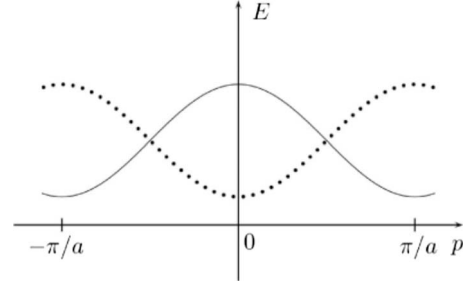


FIG. 1. Dispersion relations for electrons (dotted curve) and holes (solid curve)

$$E_e(p) = \frac{J}{2} + \frac{U}{2} + 2t \cos(pa + \pi) = \frac{J}{2} + \frac{U}{2} - 2t + ta^2 \hat{p}^2. \quad (2.25)$$

For electrons, the minima of the dispersion relation are located at $p = 2n\pi/a$, with $n \in \mathbb{Z}$. Expanding around $p = 0$, we get

$$E_e(p) = \frac{J}{2} + \frac{U}{2} - 2t + ta^2 p^2 + \mathcal{O}(p^4). \quad (2.26)$$

Again, for small momenta

$$E_e(p) = M_e + \frac{p^2}{2M'_e}. \quad (2.27)$$

Due to the $SU(2)_Q$ symmetry, the rest and kinetic masses $M_{h,e}$ and $M'_{h,e}$ of holes and electrons are identical.

C. Gap equation for magnon states

As we have discussed before, in quantum ferromagnets, the global spin rotational symmetry $SU(2)_s$ is spontaneously broken by the formation of a uniform magnetization. The ground states of these systems are invariant only under spin rotations in the subgroup $U(1)_s$. In this case, Goldstone's theorem predicts $3-1=2$ massless boson fields—the magnons—also known as ferromagnetic spin waves.

A general ansatz for an electron-hole state is given by

$$|ehp\rangle = \sum_{x,y} \exp(ipy) f(x) c_{y\downarrow}^\dagger c_{y+x\uparrow} |v\rangle. \quad (2.28)$$

Here, x is the distance between the electron and the hole and $f(x)$ is the corresponding wave function of their relative motion. Indeed, magnons are massless bound states of an electron and a hole. We now consider

$$\begin{aligned} H|ehp\rangle &= H \sum_{x,y} \exp(ipy) f(x) c_{y\downarrow}^\dagger c_{y+x\uparrow} |v\rangle \\ &= \left(\left[H, \sum_{x,y} \exp(ipy) f(x) c_{y\downarrow}^\dagger c_{y+x\uparrow} \right] \right. \\ &\quad \left. + \sum_{x,y} \exp(ipy) f(x) c_{y\downarrow}^\dagger c_{y+x\uparrow} H \right) |v\rangle. \end{aligned} \quad (2.29)$$

The last term on the right-hand side represents the vacuum energy. The electron-hole energy $E_{eh}(p)$ is given by

$$\left[H, \sum_{x,y} \exp(ipy) f(x) c_{y\downarrow}^\dagger c_{y+x\uparrow} \right] |v\rangle = E_{eh}(p) |ehp\rangle. \quad (2.30)$$

A somewhat tedious evaluation of Eq. (2.30) implies that $|ehp\rangle$ is an eigenstate only if

$$E_{eh}(p)f(x) = -t[f(x-a)(e^{ipa} - 1) + f(x+a)(e^{-ipa} - 1)] + (J+U)f(x), \quad (2.31)$$

in the generic case $x \neq 0, \pm a$, as well as

$$E_{eh}(p)f(x) = -t[f(x-a)(e^{ipa} - 1) + f(x+a)(e^{-ipa} - 1)] + \left(\frac{3}{4}J+U\right)f(x), \quad (2.32)$$

in the special case $x = \pm a$, and

$$E_{eh}(p)f(x) = -t[f(x-a)(e^{ipa} - 1) + f(x+a)(e^{-ipa} - 1)] + \left(J\frac{a^2\hat{p}^2}{2} + U\right)f(x), \quad (2.33)$$

in the special case $x=0$. These three equations represent the lattice Schrödinger equation for an electron-hole pair with wave function $f(x)$. In order to solve these equations, we transform to momentum space and obtain

$$f(q) = \frac{A \cos(qa) + B \sin(qa) + C}{E_{eh}(p) - 2t[\cos(qa) - \cos(qa - pa)] - J - U}, \quad (2.34)$$

with

$$\begin{aligned} A &= -\frac{J}{2} \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq f(q) \cos(qa), \\ B &= -\frac{J}{2} \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq f(q) \sin(qa), \\ C &= \left(J\frac{\hat{p}^2 a^2}{2} - J - U \right) \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq f(q). \end{aligned} \quad (2.35)$$

These are three coupled gap equations which must be solved self-consistently for A, B, C , and $E_{eh}(p)$. We are going to do this in the next section. The denominator in Eq. (2.34) can be rewritten as

$$E_{eh}(p) - 2t[\cos(qa) - \cos(qa - pa)] - J - U = \alpha \cos(qa) + \beta \sin(qa) + \gamma, \quad (2.36)$$

where

$$\begin{aligned} \alpha &= 2t[\cos(pa) - 1], \quad \beta = 2t \sin(pa), \\ \gamma &= E_{eh}(p) - J - U. \end{aligned} \quad (2.37)$$

D. Solution of the gap equation

Let us now solve the gap Eq. (2.35). Inserting this equation into Eq. (2.35), we obtain an eigenvalue problem with eigenvalue 1,

$$I \begin{pmatrix} A \\ B \\ C \end{pmatrix} = -\frac{J}{2} \begin{pmatrix} I_4 & I_6 & I_2 \\ I_6 & I_5 & I_3 \\ zI_2 & zI_3 & zI_1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \quad (2.38)$$

where

$$z = 2 + 2\frac{U}{J} - \hat{p}^2 a^2. \quad (2.39)$$

The quantities I_1, I_2, \dots, I_6 are integrals, which are evaluated in the Appendix. The eigenvalues of the matrix I of Eq. (2.38) are given by

$$\begin{aligned} i_1 &= -\text{sign}(\gamma) \frac{J}{2} \frac{1}{|\gamma| + s}, \\ i_2 &= -\text{sign}(\gamma) \frac{J}{4} \frac{1}{s(|\gamma| + s)} \left[|\gamma| + z(|\gamma| + s) \right. \\ &\quad \left. + \sqrt{-4s(|\gamma| + s)z + [|\gamma| + z(|\gamma| + s)]^2} \right], \\ i_3 &= -\text{sign}(\gamma) \frac{J}{4} \frac{1}{s(|\gamma| + s)} \left[|\gamma| + z(|\gamma| + s) \right. \\ &\quad \left. - \sqrt{-4s(|\gamma| + s)z + [|\gamma| + z(|\gamma| + s)]^2} \right]. \end{aligned} \quad (2.40)$$

The solutions of the gap Eq. (2.35) correspond to eigenvalues 1 in Eq. (2.38). The condition $i_1=1$ can be fulfilled only for $\gamma < 0$ and then implies

$$E_{eh}^{(1)}(p) = \frac{3}{4}J + U - \frac{4t^2\hat{p}^2 a^2}{J}. \quad (2.41)$$

Although this is the energy of an electron-hole state with total momentum p , the corresponding eigenstate does not represent a magnon because $E_{eh}^{(1)}(p)$ does not vanish for zero momentum. Similarly, the condition $i_3=1$ can be fulfilled only for $\gamma < 0$, which then implies

$$E_{eh}^{(2)}(p) = -\frac{(4J+16U)t^2}{J(3J+4U)} p^2 a^2 + \frac{3}{4}J + U + \mathcal{O}(p^4). \quad (2.42)$$

As before, this is indeed the energy of an electron-hole state, but this state is not a magnon either. Finally, (again for $\gamma < 0$) the condition $i_2=1$ implies

$$E_{eh}^{(3)}(p) = \frac{J(3J+4U) - 16t^2}{2(3J+4U)} p^2 a^2 + \mathcal{O}(p^4). \quad (2.43)$$

This energy vanishes at zero momentum. Hence, the corresponding electron-hole eigenstate can be identified as a magnon state. Indeed, the nonrelativistic dispersion relation $E_{eh}^{(3)}(p) \propto p^2$ is characteristic for ferromagnetic spin waves. In momentum space the wave function for the relative motion of the electron and hole forming the massless magnon takes the form

$$f(q) = \frac{A \cos(qa) + B \sin(qa) + C}{E_{eh}^{(3)}(p) - 2t[\cos(qa) - \cos(qa - pa)] - J - U}, \quad (2.44)$$

which turns into

$$f(x) = \mathcal{N} \left(\frac{\alpha + i\beta}{|\gamma| + s} \right)^{x/a}, \quad (2.45)$$

for $x \geq 0$, where \mathcal{N} is a normalization factor. For $x \leq 0$, one finds $f(x) = f(-x)^*$.

E. Two-hole states

Similar to the particle-hole spin-wave states, we now derive a Schrödinger equation for two-hole bound states. We make the ansatz

$$|hhp\rangle = \sum_{x,y} \exp(ipy) g(x) c_{y\uparrow} c_{y+x\uparrow} |v\rangle,$$

$$g(-x) = -g(x) \exp(-ipx). \quad (2.46)$$

The antisymmetry condition $g(-x) = -g(x) \exp(-ipx)$ follows from the Pauli principle. In complete analogy to the particle-hole states, one derives the Schrödinger equation

$$E_{hh}(p)g(x) = t[g(x+a)(1 + e^{-ipa}) + g(x-a)(1 + e^{ipa})] + (J+U)g(x), \quad (2.47)$$

for a generic situation with $x \neq 0, \pm a$. In the special case $x = \pm a$, one obtains

$$E_{hh}(p)g(x) = t[g(x+a)(1 + e^{-ipa}) + g(x-a)(1 + e^{ipa})] + \left(\frac{3}{4}J + U\right)g(x), \quad (2.48)$$

while for $x=0$, the Schrödinger equation takes the form

$$E_{hh}(p)g(x) = t[g(x+a)(1 + e^{-ipa}) + g(x-a)(1 + e^{ipa})]. \quad (2.49)$$

Going to momentum space, one obtains the gap equation

$$A = -\frac{J}{2} \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq g(q) \cos(qa),$$

$$B = -\frac{J}{2} \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq g(q) \sin(qa), \quad (2.50)$$

with

$$g(q) = \frac{A \cos(qa) + B \sin(qa)}{E_{hh}(p) - 2t[\cos(qa) - \cos(qa - pa)] - J - U}. \quad (2.51)$$

The antisymmetry condition $g(-x) = -g(x) \exp(-ipx)$ implies $g(-q) = -g(p+q)$. Imposing this condition on the gap equation leads to

$$\begin{pmatrix} -\cos(pa) & -\sin(pa) \\ -\sin(pa) & \cos(pa) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\Rightarrow \cos\left(\frac{pa}{2}\right)A = -\sin\left(\frac{pa}{2}\right)B. \quad (2.52)$$

We now introduce C , such that

$$A = C \sin\left(\frac{pa}{2}\right), \quad B = -C \cos\left(\frac{pa}{2}\right). \quad (2.53)$$

The gap equation can thus be written as

$$C = \frac{J}{2} \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq g(q) \sin\left(qa - \frac{pa}{2}\right), \quad (2.54)$$

with

$$g(q) = \frac{-C \sin(qa - pa/2)}{E_{hh}(p) - 2t[\cos(pa - qa) + \cos(qa)] - J - U}. \quad (2.55)$$

Let us now solve the gap equation. Inserting Eq. (2.55) into Eq. (2.54) yields

$$1 = -\frac{J}{2} \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq \frac{\sin^2(qa)}{E_{hh}(p) - 4t \cos(qa) \cos(pa/2) - J - U}. \quad (2.56)$$

Using

$$\frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq \frac{\sin^2(qa)}{\alpha \cos(qa) + \gamma} = \text{sign}(\gamma) \frac{1}{|\gamma| + \sqrt{\gamma^2 - \alpha^2}}, \quad (2.57)$$

with

$$\gamma = E_{hh}(p) - J - U, \quad \alpha = -4t \cos\left(\frac{pa}{2}\right), \quad (2.58)$$

for $\gamma < 0$, one thus obtains

$$\frac{J}{2} = J + U - E_{hh}(p) + \sqrt{[E_{hh}(p) - J - U]^2 - 16t^2 \cos^2(pa/2)}. \quad (2.59)$$

Squaring this equation, we find the two-hole energy

$$E_{hh}(p) = \frac{3}{4}J + U - \frac{16t^2}{J} + \frac{4t^2 p^2 a^2}{J} + \mathcal{O}(p^4). \quad (2.60)$$

From this expression, we read off the total rest mass of the two-hole bound state as

$$M_{hh} = \frac{3}{4}J + U - \frac{16t^2}{J}, \quad (2.61)$$

while the corresponding kinetic mass is given by

$$M'_{hh} = \frac{J}{8t^2 a^2}. \quad (2.62)$$

The binding energy of the two-hole state hence takes the form

$$E_B = 2M_h - M_{hh} = J \left(\frac{1}{2} - \frac{4t}{J} \right)^2. \quad (2.63)$$

As we will see below, the two-hole bound-state wave function is normalizable only for $0 < t < J/8$, while for $t > J/8$ the two-hole bound state disappears.

Let us now consider the wave function in coordinate space. For $\gamma < 0$, we obtain

$$g(x) = C_0 \exp\left(\frac{ipx}{2}\right) \left[-8 \frac{t}{J} \cos\left(\frac{pa}{2}\right) \right]^{x/a}, \quad (2.64)$$

where C_0 is a normalization constant. The expression Eq. (2.64) indeed satisfies the antisymmetry condition $g(-x) = -g(x)\exp(-ipx)$.

Let us also consider scattering states of two holes described by the ansatz

$$g(x) = \tilde{A} \exp(iqx) + \tilde{B} \exp(-iqx). \quad (2.65)$$

For $x \neq 0, \pm a$ and $p=0$, the Schrödinger equation is given by

$$2t[g(x+a) + g(x-a)] + (J+U)g(x) = Eg(x). \quad (2.66)$$

Inserting the ansatz of Eq. (2.65) into Eq. (2.66) leads to

$$E = 4t \cos(qa) + J + U, \quad (2.67)$$

while the amplitudes are given by

$$\begin{aligned} \tilde{A} &= \frac{J}{16t} + i \frac{J \cos(qa) + 8t}{16t \sin(qa)}, \\ \tilde{B} &= \frac{J}{16t} - i \frac{J \cos(qa) + 8t}{16t \sin(qa)} = \tilde{A}^*. \end{aligned} \quad (2.68)$$

This implies that the ansatz of Eq. (2.65) is purely real. Finally, we obtain

$$\frac{\tilde{A}}{\tilde{B}} = \frac{\sin(qa) + i[\cos(qa) + 8t/J]}{\sin(qa) - i[\cos(qa) + 8t/J]}. \quad (2.69)$$

Later, we will compare this result with the corresponding one obtained in the effective field theory.

III. CONSTRUCTION OF THE EFFECTIVE FIELD THEORY FOR THE DOPED FERROMAGNET

The effective field theory we are going to discuss now captures the low-energy physics of the underlying microscopic system, order by order in a systematic low-energy expansion. The effective field theory is constructed in complete analogy to the corresponding cases of hole- or electron-doped antiferromagnets.^{20,21,23}

A. Symmetry properties of magnon fields

In this section, we are going to investigate the symmetries of magnon fields. At the beginning of Sec. II, we have studied the symmetries of the microscopic model describing ferromagnetism. The effective field theory must share the sym-

metries of the underlying microscopic system. Therefore, we now construct magnon fields and discuss how they transform under those symmetries.

As we have mentioned in Sec. II, in a quantum ferromagnet the global spin rotation symmetry $G = SU(2)_s$ is spontaneously broken by the formation of a uniform magnetization. The ground state of these systems is invariant only under spin rotations in the unbroken subgroup $H = U(1)_s$. As a consequence of the spontaneous symmetry breaking, there are two massless Goldstone boson fields, leading to the ferromagnetic spin wave or magnon. We already discussed magnons in the microscopic model, where we calculated the dispersion relation in Eq. (2.43). In the effective field theory, the direction of the magnetization is described by a unit-vector field

$$\vec{e}(x) = (e_1(x), e_2(x), e_3(x)) \in S^2, \quad \vec{e}(x)^2 = 1, \quad (3.1)$$

in the coset space $G/H = SU(2)_s / U(1)_s = S^2$, where $x = (x_1, t)$ is a point in (1+1)-dimensional Euclidean space-time.

Beyond the $O(3)$ vector representation $\vec{e}(x)$, it is useful to introduce an alternative $CP(1)$ representation of the magnon field using 2×2 Hermitean projection matrices $P(x)$ that obey

$$P(x)^\dagger = P(x), \quad \text{Tr } P(x) = 1, \quad P(x)^2 = P(x), \quad (3.2)$$

and are given by

$$\begin{aligned} P(x) &= \frac{1}{2} (1 + \vec{e}(x) \cdot \vec{\sigma}) \\ &= \frac{1}{2} \begin{pmatrix} 1 + e_3(x) & e_1(x) - ie_2(x) \\ e_1(x) + ie_2(x) & 1 - e_3(x) \end{pmatrix}. \end{aligned} \quad (3.3)$$

The first symmetry we encountered in Sec. II was the global spin rotation symmetry $SU(2)_s$ under which the magnon field transforms as

$$P(x)' = g P(x) g^\dagger. \quad (3.4)$$

Note that the magnon field $P(x)$ is invariant under the Abelian and non-Abelian fermion number symmetries $U(1)_Q$ and $SU(2)_Q$, i.e.,

$$\tilde{Q} P(x) = P(x). \quad (3.5)$$

Unlike in an antiferromagnet, in a ferromagnet the order parameter $\vec{e}(x)$ is invariant under the displacement symmetry D , i.e.,

$${}^D \vec{e}(x) = \vec{e}(x) \Rightarrow {}^D P(x) = P(x). \quad (3.6)$$

Under the spatial reflection R , which turns the point $x = (x_1, t)$ into the reflected point $Rx = (-x_1, t)$, the magnon field transforms as

$${}^R P(x) = P(Rx). \quad (3.7)$$

Another important symmetry is time reversal T , which turns x into $Tx = (x_1, -t)$. The spin transforms like the orbital angular momentum \vec{L} of a particle. The momentum \vec{p} changes sign under time reversal and so does \vec{L} , i.e., ${}^T \vec{L} = -\vec{L}$. Consequently, under T the magnetization vector (which is a sum of microscopic spins) transforms as

$${}^T\vec{e}(x) = -\vec{e}(Tx) \Rightarrow {}^T P(x) = 1 - P(Tx). \quad (3.8)$$

B. Effective action for magnons

Since the low-energy physics is dominated by terms with the smallest possible number of derivatives, we construct an effective Lagrangian according to a systematic derivative expansion. All terms in the Lagrangian must be invariant under the symmetry transformations considered in the previous section. In contrast to an antiferromagnet, a ferromagnet has a conserved order parameter—the total spin. In the effective theory, this manifests itself by the presence of a Wess-Zumino term, which gives rise to a nonrelativistic magnon dispersion relation.³³ Indeed, as we have seen from the calculations in the microscopic model, the ferromagnet has a nonrelativistic spectrum, i.e., $E \sim p^2$. The leading order Euclidean effective action for an undoped ferromagnet derived in Ref. 33 takes the form

$$S[\vec{e}] = \int dx_1 \int_0^\beta dt \frac{\rho_s}{2} \partial_1 \vec{e} \cdot \partial_1 \vec{e} + S_{\text{WZ}}[\vec{e}], \quad (3.9)$$

with ρ_s being the spin stiffness. The Wess-Zumino term is given by

$$S_{\text{WZ}}[\vec{e}] = -im \int dx_1 \int_0^\beta dt \int_0^1 d\tau \vec{e} \cdot (\partial_t \vec{e} \times \partial_\tau \vec{e}). \quad (3.10)$$

Here, m is the magnetization density. This term contains only one temporal derivative and hence leads to a nonrelativistic dispersion relation. The coordinates t and τ parameterize a disk or two-dimensional hemisphere H^2 , which is bounded by the compactified Euclidean time interval S^1 . The magnon field $\vec{e}(x)$ at physical space-time points $x \in \mathbb{R} \times S^1$ is extended to a field $\vec{e}(x_1, t, \tau)$ in the three-dimensional domain $(x_1, t, \tau) \in \mathbb{R} \times H^2$. The integrand of the Wess-Zumino term is a total derivative and hence only receives contributions from the boundary, which coincides with the physical space-time where $\vec{e}(x, \tau=1) = \vec{e}(x)$. A possible extrapolation of the physical magnon field into the additional dimension with $\vec{e}(x, \tau=0) = (0, 0, 1)$ is given by

$$\begin{aligned} e_1(x, \tau) &= \tau e_1(x), & e_2(x, \tau) &= \tau e_2(x), \\ e_3(x, \tau) &= \sqrt{1 - e_1(x, \tau)^2 - e_2(x, \tau)^2}. \end{aligned} \quad (3.11)$$

The action of Eq. (3.9) enters the Euclidean path integral

$$Z = \int \mathcal{D}\vec{e} \exp(-S[\vec{e}]), \quad (3.12)$$

which should depend only on the physical magnon field and not on a particular extrapolation into the additional dimension. In order to show that this is indeed the case, we compare two arbitrary extrapolations $\vec{e}^{(1)}(x, \tau)$ and $\vec{e}^{(2)}(x, \tau)$ and we consider the difference between the two corresponding Wess-Zumino terms

$$\begin{aligned} S_{\text{WZ}}[\vec{e}^{(1)}] - S_{\text{WZ}}[\vec{e}^{(2)}] &= -im \int dx_1 \int_{H^2} dt d\tau \vec{e}^{(1)} \cdot (\partial_t \vec{e}^{(1)} \times \partial_\tau \vec{e}^{(1)}) \\ &\quad + im \int dx_1 \int_{H^2} dt d\tau \vec{e}^{(2)} \cdot (\partial_t \vec{e}^{(2)} \times \partial_\tau \vec{e}^{(2)}) \\ &= -im \int dx_1 \int_{S^2} dt d\tau \vec{e} \cdot (\partial_t \vec{e} \times \partial_\tau \vec{e}). \end{aligned} \quad (3.13)$$

The two extrapolations $\vec{e}^{(1)}$ and $\vec{e}^{(2)}$ over the two hemispheres H^2 (which are differently oriented due to the minus sign between the Wess-Zumino terms) are combined to an extrapolation \vec{e} over an entire compact sphere S^2 . We now use the fact that

$$\frac{1}{4\pi} \int_{S^2} dt d\tau \vec{e} \cdot (\partial_t \vec{e} \times \partial_\tau \vec{e}) = n \in \mathbb{Z} \quad (3.14)$$

is the integer winding number of the field \vec{e} which maps S^2 (parameterized by t and τ) into the order parameter sphere S^2 . Indeed, the corresponding second homotopy group is given by $\Pi_2[S^2] = \mathbb{Z}$. Hence, the extrapolation ambiguity is given by

$$S_{\text{WZ}}[\vec{e}^{(1)}] - S_{\text{WZ}}[\vec{e}^{(2)}] = -im \int dx_1 4\pi n. \quad (3.15)$$

Since m is the magnetization density,

$$M = m \int dx_1 \quad (3.16)$$

is the total spin of the entire magnet and hence an integer or a half-integer. Since

$$\exp(-S_{\text{WZ}}[\vec{e}^{(1)}] + S_{\text{WZ}}[\vec{e}^{(2)}]) = \exp(4\pi i M n) = 1, \quad (3.17)$$

the factor $\exp(-S[\vec{e}])$ that enters the path integral of Eq. (3.12) is thus unambiguously defined, irrespective of the arbitrarily chosen extrapolation $\vec{e}(x, \tau)$.

In the $\mathbb{C}P(1)$ representation, the leading order low-energy Euclidean action takes the form

$$\begin{aligned} S[P] &= \int dx_1 \int_0^\beta dt \text{Tr}[\rho_s \partial_1 P \partial_1 P \\ &\quad + 2m \int_0^1 d\tau P(\partial_t P \partial_\tau P - \partial_\tau P \partial_t P)]. \end{aligned} \quad (3.18)$$

From Eq. (3.9), one can derive the Landau-Lifshitz equation for spin waves in a ferromagnet⁵⁶

$$\rho_s \vec{e} \times \partial_t^2 \vec{e} = m \partial_t \vec{e}. \quad (3.19)$$

We assume a magnetization in the three-direction with small perturbations in the one- and two-directions, i.e.,

$$\vec{e}(x) = \left(\frac{m_1(x)}{\sqrt{\rho_s}}, \frac{m_2(x)}{\sqrt{\rho_s}}, 1 \right) + \mathcal{O}(m^2). \quad (3.20)$$

Expanding up to linear powers in the magnon fluctuations m_1 and m_2 , one obtains the equation

$$i\partial_t(m_1 + im_2) = -\frac{\rho_s}{m}\partial_1^2(m_1 + im_2), \quad (3.21)$$

which implies the nonrelativistic magnon dispersion relation

$$E_m(p) = \frac{\rho_s p^2}{m}. \quad (3.22)$$

C. Determination of the low-energy parameters

At this point, we can match the low-energy parameters m and ρ_s of the effective field theory to the coupling constants t , J , and U of the underlying microscopic system. First of all, in the ferromagnetic ground state all spins are up, and hence the magnetization density is given by

$$m = \frac{1}{2a}. \quad (3.23)$$

The magnon dispersion relation obtained in the microscopic model was given by

$$E_{eh}^{(3)}(p) = \frac{J(3J+4U) - 16t^2}{2(3J+4U)} p^2 a^2 + \mathcal{O}(p^4). \quad (3.24)$$

Identifying $E_{eh}^{(3)}(p)$ with $E_m(p)$, we read off the value

$$\rho_s = \frac{J(3J+4U) - 16t^2}{4(3J+4U)} a, \quad (3.25)$$

for the spin stiffness. For $t=0$, the microscopic model reduces to the Heisenberg model and the spin stiffness takes the familiar value $\rho_s = Ja/4$. It should be noted that the ferromagnetic vacuum becomes unstable when $16t^2 > J(3J+4U)$.

D. Nonlinear realization of the $SU(2)_s$ spin symmetry

In order to couple electron or hole fields to the order parameter, a nonlinear realization of the $SU(2)_s$ symmetry has been constructed in Refs. 20, 21, and 23. A local transformation $h(x) \in U(1)_s$ is then constructed from the global transformation $g \in SU(2)_s$ as well as from the local magnon field $P(x)$ as follows. First, one diagonalizes the magnon field by a unitary transformation $u(x) \in SU(2)_s$, i.e.,

$$u(x)P(x)u(x)^\dagger = \frac{1}{2}(1 + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_{11}(x) \geq 0. \quad (3.26)$$

Note that, due to its projector properties, $P(x)$ has eigenvalues 0 and 1. In order to make $u(x)$ uniquely defined, we demand that the element $u_{11}(x)$ is real and non-negative. Otherwise, the diagonalizing matrix $u(x)$ would be defined only up to a $U(1)_s$ phase. Using Eq. (3.3) and spherical coordinates for $\vec{e}(x)$, i.e.,

$$\vec{e}(x) = (\sin \theta(x) \cos \varphi(x), \sin \theta(x) \sin \varphi(x), \cos \theta(x)), \quad (3.27)$$

one obtains

$$\begin{aligned} u(x) &= \frac{1}{\sqrt{2(1+e_3(x))}} \begin{pmatrix} 1+e_3(x) & e_1(x) - ie_2(x) \\ -e_1(x) - ie_2(x) & 1+e_3(x) \end{pmatrix} \\ &= \begin{pmatrix} \cos[\theta(x)/2] & \sin[\theta(x)/2] \exp[-i\varphi(x)] \\ -\sin[\theta(x)/2] \exp[i\varphi(x)] & \cos[\theta(x)/2] \end{pmatrix}. \end{aligned} \quad (3.28)$$

Under a global $SU(2)_s$ transformation g , the diagonalizing field $u(x)$ transforms as

$$u(x)' = h(x)u(x)g^\dagger, \quad u_{11}(x)' \geq 0, \quad (3.29)$$

which implicitly defines the nonlinear symmetry transformation

$$h(x) = \exp[i\alpha(x)\sigma_3] = \begin{pmatrix} \exp[i\alpha(x)] & 0 \\ 0 & \exp[-i\alpha(x)] \end{pmatrix} \in U(1)_s. \quad (3.30)$$

The transformation $h(x)$ is uniquely defined since we demand that $u_{11}(x)'$ is again real and non-negative.

Since in a ferromagnet the order parameter is invariant under the displacement symmetry D , we have

$${}^D u(x) = u(x). \quad (3.31)$$

In order to couple electrons and holes to the magnons, it is necessary to introduce the anti-Hermitian traceless field

$$v_\mu(x) = u(x)\partial_\mu u(x)^\dagger, \quad (3.32)$$

which under $SU(2)_s$ transforms as

$$v_\mu(x)' = h(x)u(x)g^\dagger \partial_\mu [gu(x)^\dagger h(x)^\dagger] = h(x)[v_\mu(x) + \partial_\mu]h(x)^\dagger. \quad (3.33)$$

Since the field $v_\mu(x)$ is traceless, it can be written as a linear combination of the Pauli matrices σ_a

$$v_\mu(x) = iv_\mu^a(x)\sigma_a, \quad a \in \{1, 2, 3\}, \quad v_\mu^a(x) \in \mathbb{R}. \quad (3.34)$$

The factor i is needed to make $v_\mu(x)$ anti-Hermitian. Introducing

$$v_\mu^\pm(x) = v_\mu^1(x) \mp iv_\mu^2(x), \quad (3.35)$$

we write

$$v_\mu(x) = i \begin{pmatrix} v_\mu^3(x) & v_\mu^+(x) \\ v_\mu^-(x) & -v_\mu^3(x) \end{pmatrix}. \quad (3.36)$$

This leads to the transformation laws for $v_\mu^3(x)$

$$SU(2)_s: v_\mu^3(x)' = v_\mu^3(x) - \partial_\mu \alpha(x),$$

$$U(1)_Q: v_\mu^3(x)' = v_\mu^3(x),$$

$$D: v_\mu^3(x)' = v_\mu^3(x),$$

$$\begin{aligned} R: {}^R v_1^3(x) &= -v_1^3(Rx), & {}^R v_t^3(x) &= v_t^3(Rx), \\ T: {}^T v_1^3(x) &= -v_1^3(Tx), & {}^T v_t^3(x) &= v_t^3(Tx), \end{aligned} \quad (3.37)$$

as well as for $v_\mu^\pm(x)$

$$\begin{aligned} SU(2)_s: v_\mu^\pm(x)' &= \exp[\pm 2i\alpha(x)]v_\mu^\pm(x), \\ U(1)_Q: v_\mu^\pm(x) &= v_\mu^\pm(x), \\ D: v_\mu^\pm(x) &= v_\mu^\pm(x), \\ R: {}^R v_1^\pm(x) &= -v_1^\pm(Rx), & {}^R v_t^\pm(x) &= v_t^\pm(Rx), \\ T: {}^T v_1^\pm(x) &= -v_1^\pm(Tx), & {}^T v_t^\pm(x) &= v_t^\pm(Tx). \end{aligned} \quad (3.38)$$

E. Microscopic operators in a magnon background field

In the context of the effective field theory, until now we have only discussed magnons, which correspond to states at half-filling in the microscopic model. In this section, we will begin to include doped electrons and holes. For this purpose, we must establish a connection between the microscopic degrees of freedom and the low-energy effective fields describing electrons or holes. Following Refs. 20, 21, and 23, we now discuss how this connection is established. It is a virtue of the completely analytically controlled ferromagnetic case that this connection can be tested rigorously.

As discussed in detail in Ref. 20, 21, and 23 in order to define new operators Ψ_x^A and Ψ_x^B , it is useful to introduce the matrix-valued fermion operator C_x . We have already used the operator C_x in Eq. (2.14) to rewrite the Hamiltonian in a manifestly $SU(2)_s$, $SU(2)_Q$, D -, and R -invariant form. Now, we write

$$\begin{aligned} \Psi_x^A &= u(x)C_x = u(x) \begin{pmatrix} c_{x\uparrow} & c_{x\downarrow}^\dagger \\ c_{x\downarrow} & -c_{x\uparrow}^\dagger \end{pmatrix} = \begin{pmatrix} \psi_{x+}^A & \psi_{x-}^{A\dagger} \\ \psi_{x-}^A & -\psi_{x+}^{A\dagger} \end{pmatrix}, & x \in A, \\ \Psi_x^B &= u(x)C_x = u(x) \begin{pmatrix} c_{x\uparrow} & -c_{x\downarrow}^\dagger \\ c_{x\downarrow} & c_{x\uparrow}^\dagger \end{pmatrix} = \begin{pmatrix} \psi_{x+}^B & -\psi_{x-}^{B\dagger} \\ \psi_{x-}^B & \psi_{x+}^{B\dagger} \end{pmatrix}, & x \in B. \end{aligned} \quad (3.39)$$

Note that Ψ denotes a matrix while ψ denotes a matrix element. The new lattice operators inherit their transformation properties from the operators of the microscopic model, i.e., we use the transformation properties of C_x discussed in Sec. II. It should be noted that here the continuum field $u(x)$ is evaluated only at discrete lattice points x . According to Eqs. (2.11) and (3.29), under the $SU(2)_s$ symmetry, one obtains

$$\Psi_x^{A,B'} = u(x)' C_x' = h(x)u(x)g^\dagger g C_x = h(x)\Psi_x^{A,B}. \quad (3.40)$$

In components, this relation takes the form

$$\psi_{x\pm}^{A,B'} = \exp[\pm i\alpha(x)]\psi_{x\pm}^{A,B}. \quad (3.41)$$

Similarly, under the $SU(2)_Q$ symmetry, one obtains

$$\tilde{Q}\Psi_x^{A,B} = \tilde{Q}u(x)\tilde{Q}C_x = u(x)C_x\Omega^T = \Psi_x^{A,B}\Omega^T. \quad (3.42)$$

Here, we have used the fact that $u(x)$ is invariant under the fermion number symmetries $U(1)_Q$ and $SU(2)_Q$, i.e., $\tilde{Q}u(x) = u(x)$. In particular, under the $U(1)_Q$ subgroup of $SU(2)_Q$, the components transform as

$${}^Q\psi_{x\pm}^{A,B} = \exp(i\omega)\psi_{x\pm}^{A,B}. \quad (3.43)$$

Under the displacement symmetry, we obtain

$${}^D\Psi_x^{A,B} = {}^D u(x){}^D C_x^{A,B} = u(x+a)C_{x+a}^{B,A}\sigma_3 = \Psi_{x+a}^{B,A}. \quad (3.44)$$

Expressed in terms of components, this implies

$${}^D\psi_{x\pm}^{A,B} = \psi_{x+a\pm}^{B,A}. \quad (3.45)$$

F. Effective fields for charge carriers

In the low-energy effective field theory, we will use a Euclidean path integral description instead of the Hamiltonian description used in the microscopic model. The lattice operators $\psi_{x\pm}^{A,B}$ and $\psi_{x\pm}^{A,B\dagger}$ are then replaced by Grassmann numbers $\psi_\pm^{A,B}(x)$ and $\psi_\pm^{A,B\dagger}(x)$, which are completely independent of each other. Therefore, in the effective field theory the electron and hole fields are represented by eight independent Grassmann numbers $\psi_\pm^{A,B}(x)$ and $\psi_\pm^{A,B\dagger}(x)$, which can be combined to

$$\begin{aligned} \Psi^A(x) &= \begin{pmatrix} \psi_+^A(x) & \psi_-^{A\dagger}(x) \\ \psi_-^A(x) & -\psi_+^{A\dagger}(x) \end{pmatrix}, \\ \Psi^B(x) &= \begin{pmatrix} \psi_+^B(x) & -\psi_-^{B\dagger}(x) \\ \psi_-^B(x) & \psi_+^{B\dagger}(x) \end{pmatrix}. \end{aligned} \quad (3.46)$$

For notational convenience, we also introduce the fields

$$\begin{aligned} \Psi^{A\dagger}(x) &= \begin{pmatrix} \psi_+^{A\dagger}(x) & \psi_-^{A\dagger}(x) \\ \psi_-^A(x) & -\psi_+^A(x) \end{pmatrix}, \\ \Psi^{B\dagger}(x) &= \begin{pmatrix} \psi_+^{B\dagger}(x) & \psi_-^{B\dagger}(x) \\ -\psi_-^B(x) & \psi_+^B(x) \end{pmatrix}. \end{aligned} \quad (3.47)$$

We should note that $\Psi^{A,B\dagger}(x)$ is not independent of $\Psi^{A,B}(x)$, since both contain the same Grassmann fields $\psi_\pm^{A,B}(x)$ and $\psi_\pm^{A,B\dagger}(x)$. It should also be pointed out that the continuum fields of the low-energy effective theory cannot be derived explicitly from the lattice operators of the microscopic model. Still, the Grassmann fields $\Psi^{A,B}(x)$ describing electrons and holes in the low-energy effective theory transform just like the lattice operators $\Psi_x^{A,B}$ discussed before. In contrast to the lattice operators, the fields $\Psi^{A,B}(x)$ are defined in the continuum. Hence, under the displacement symmetry D one no longer distinguishes between the points x and $x+a$.

We now list the transformation properties of the effective fields under the various symmetries, which can be derived using the transformation properties discussed above

$$SU(2)_s: \Psi^{A,B}(x)' = h(x)\Psi^{A,B}(x), \quad \Psi^{A,B\dagger}(x)' = \Psi^{A,B\dagger}(x)h(x)^\dagger,$$

$$\begin{aligned}
SU(2)_Q: \tilde{Q}\Psi^{A,B}(x) &= \Psi^{A,B}(x)\Omega^T, & \tilde{Q}\Psi^{A,B^\dagger}(x) &= \Omega^T \Psi^{A,B^\dagger}(x), \\
D: {}^D\Psi^{A,B}(x) &= \Psi^{B,A}(x)\sigma_3, & {}^D\Psi^{A,B^\dagger}(x) &= \sigma_3 \Psi^{B,A^\dagger}(x), \\
R: {}^R\Psi^{A,B}(x) &= \Psi^{A,B}(Rx), & {}^R\Psi^{A,B^\dagger}(x) &= \Psi^{A,B^\dagger}(Rx), \\
T: {}^T\Psi^{A,B}(x) &= -[\Psi^{A,B^\dagger}(Tx)^T]\sigma_3, \\
{}^T\Psi^{A,B^\dagger}(x) &= \sigma_3[\Psi^{A,B}(Tx)^T]. \tag{3.48}
\end{aligned}$$

Note, that an upper index T on the right denotes transpose, while on the left it denotes time reversal. In components, the symmetry transformations read

$$\begin{aligned}
SU(2)_s: \psi_{\pm}^{A,B}(x)' &= \exp[\pm i\alpha(x)]\psi_{\pm}^{A,B}(x), \\
\psi_{\pm}^{A,B^\dagger}(x)' &= \exp[\mp i\alpha(x)]\psi_{\pm}^{A,B^\dagger}(x), \\
U(1)_Q: {}^Q\psi_{\pm}^{A,B}(x) &= \exp(i\omega)\psi_{\pm}^{A,B}(x), \\
{}^Q\psi_{\pm}^{A,B^\dagger}(x) &= \exp(-i\omega)\psi_{\pm}^{A,B^\dagger}(x), \\
D: {}^D\psi_{\pm}^{A,B}(x) &= \psi_{\pm}^{B,A}(x), & {}^D\psi_{\pm}^{A,B^\dagger}(x) &= \psi_{\pm}^{B,A^\dagger}(x), \\
R: {}^R\psi_{\pm}^{A,B}(x) &= \psi_{\pm}^{A,B}(Rx), & {}^R\psi_{\pm}^{A,B^\dagger}(x) &= \psi_{\pm}^{A,B^\dagger}(Rx), \\
T: {}^T\psi_{\pm}^{A,B}(x) &= -\psi_{\pm}^{A,B^\dagger}(Tx), & {}^T\psi_{\pm}^{A,B^\dagger}(x) &= \psi_{\pm}^{A,B}(Tx). \tag{3.49}
\end{aligned}$$

G. Fermion fields in momentum space pockets

As we know from the microscopic model, the electrons live in a momentum space pocket around $p=0$ and have a spin opposite to the total magnetization, while the holes live in a pocket around $p=\pi/a$ and have a spin parallel to the magnetization. In order to describe these low-energy fermion degrees of freedom, we perform a discrete Fourier transform from the sublattice indices A and B to the momentum space pocket indices 0 and π . Again, this is in complete analogy to the antiferromagnetic case discussed in Refs. 21 and 23,

$$\begin{aligned}
\psi_{-}^0(x) &= \frac{1}{\sqrt{2}}[\psi_{-}^A(x) + \psi_{-}^B(x)], \\
\psi_{-}^{0^\dagger}(x) &= \frac{1}{\sqrt{2}}[\psi_{-}^{A^\dagger}(x) + \psi_{-}^{B^\dagger}(x)], \\
\psi_{+}^{\pi}(x) &= \frac{1}{\sqrt{2}}[\psi_{+}^A(x) - \psi_{+}^B(x)], \\
\psi_{+}^{\pi^\dagger}(x) &= \frac{1}{\sqrt{2}}[\psi_{+}^{A^\dagger}(x) - \psi_{+}^{B^\dagger}(x)]. \tag{3.50}
\end{aligned}$$

We obtain the transformation rules

$$SU(2)_s: \psi_{-}^0(x)' = \exp[-i\alpha(x)]\psi_{-}^0(x),$$

$$\begin{aligned}
\psi_{-}^{0^\dagger}(x)' &= \exp[i\alpha(x)]\psi_{-}^{0^\dagger}(x), \\
\psi_{+}^{\pi}(x)' &= \exp[i\alpha(x)]\psi_{+}^{\pi}(x), \\
\psi_{+}^{\pi^\dagger}(x)' &= \exp[-i\alpha(x)]\psi_{+}^{\pi^\dagger}(x), \\
U(1)_Q: {}^Q\psi_{-}^0(x) &= \exp(i\omega)\psi_{-}^0(x), \\
{}^Q\psi_{-}^{0^\dagger}(x) &= \exp(-i\omega)\psi_{-}^{0^\dagger}(x), \\
{}^Q\psi_{+}^{\pi}(x) &= \exp(i\omega)\psi_{+}^{\pi}(x), \\
{}^Q\psi_{+}^{\pi^\dagger}(x) &= \exp(-i\omega)\psi_{+}^{\pi^\dagger}(x), \\
D: {}^D\psi_{-}^0(x) &= \psi_{-}^0(x), \\
{}^D\psi_{-}^{0^\dagger}(x) &= \psi_{-}^{0^\dagger}(x), \\
{}^D\psi_{+}^{\pi}(x) &= -\psi_{+}^{\pi}(x), \\
{}^D\psi_{+}^{\pi^\dagger}(x) &= -\psi_{+}^{\pi^\dagger}(x), \\
R: {}^R\psi_{-}^0(x) &= \psi_{-}^0(Rx), \\
{}^R\psi_{-}^{0^\dagger}(x) &= \psi_{-}^{0^\dagger}(Rx), \\
{}^R\psi_{+}^{\pi}(x) &= \psi_{+}^{\pi}(Rx), \\
{}^R\psi_{+}^{\pi^\dagger}(x) &= \psi_{+}^{\pi^\dagger}(Rx), \\
T: {}^T\psi_{-}^0(x) &= -\psi_{-}^{0^\dagger}(Tx), \\
{}^T\psi_{-}^{0^\dagger}(x) &= \psi_{-}^0(Tx), \\
{}^T\psi_{+}^{\pi}(x) &= -\psi_{+}^{\pi^\dagger}(Tx), \\
{}^T\psi_{+}^{\pi^\dagger}(x) &= \psi_{+}^{\pi}(Tx). \tag{3.51}
\end{aligned}$$

The effective Lagrangian to be constructed in the next section must be invariant under all these symmetry transformations as well as under the $SU(2)_Q$ transformations. The latter do not have a simple form in terms of the momentum space pocket fields (and have thus not been listed here), but they follow from Eq. (3.48).

H. Effective action for charge carriers

We now construct the leading terms of the effective action for charge carriers, which must be invariant under the symmetries $SU(2)_s$, $SU(2)_Q$, D , R , and T . We use the indices n_t , n_x , and n_ψ in a contribution to the Lagrangian $\mathcal{L}_{n_t, n_x, n_\psi}$ to denote the number of temporal derivatives n_t , the number of spatial derivatives n_x , and the number of fermion fields n_ψ . The effective Lagrangian then takes the form

$$\mathcal{L} = \sum_{n_t, n_x, n_\psi} \mathcal{L}_{n_t, n_x, n_\psi}. \tag{3.52}$$

The mass term is given by

$$\mathcal{L}_{0,0,2} = M(\psi_+^{\pi\dagger}\psi_+^\pi - \psi_-^{0\dagger}\psi_-^0). \quad (3.53)$$

In order to express the terms with spatial or temporal derivatives, we introduce the covariant derivative D_μ , which acts as

$$\begin{aligned} D_\mu\psi_-^0(x) &= [\partial_\mu - iv_\mu^3(x)]\psi_-^0(x), \\ D_\mu\psi_-^{0\dagger}(x) &= [\partial_\mu + iv_\mu^3(x)]\psi_-^{0\dagger}(x), \\ D_\mu\psi_+^\pi(x) &= [\partial_\mu + iv_\mu^3(x)]\psi_+^\pi(x), \\ D_\mu\psi_+^{\pi\dagger}(x) &= [\partial_\mu - iv_\mu^3(x)]\psi_+^{\pi\dagger}(x). \end{aligned} \quad (3.54)$$

Using the transformation laws of $v_\mu^3(x)$ listed in Eq. (3.37), one arrives at the terms

$$\mathcal{L}_{1,0,2} = \psi_+^{\pi\dagger}D_t\psi_+^\pi + \psi_-^{0\dagger}D_t\psi_-^0, \quad (3.55)$$

as well as

$$\begin{aligned} \mathcal{L}_{0,2,2} &= \frac{1}{2M'}(D_1\psi_+^{\pi\dagger}D_1\psi_+^\pi - D_1\psi_-^{0\dagger}D_1\psi_-^0) \\ &+ N(\psi_-^{0\dagger}v_1^-v_1^+\psi_-^0 + \psi_+^{\pi\dagger}v_1^+v_1^-\psi_+^\pi). \end{aligned} \quad (3.56)$$

In contrast to an antiferromagnet, there is no fermion-single-magnon vertex. Instead, all vertices contain at least two magnons. This implies that the fermion-magnon interactions in a doped ferromagnet are of higher order than in an antiferromagnet.

Using the algebraic manipulation program FORM, we have also constructed all terms involving four fermion fields and up to one temporal or two spatial derivatives. They are not very illuminating and we thus do not list them here. Instead we just concentrate on the fermionic Lagrangian in the two-hole sector, which will be used later and which takes the form

$$\begin{aligned} \mathcal{L} &= M\psi_+^{\pi\dagger}\psi_+^\pi + \psi_+^{\pi\dagger}D_t\psi_+^\pi + \frac{1}{2M'}D_1\psi_+^{\pi\dagger}D_1\psi_+^\pi + N\psi_+^{\pi\dagger}v_1^+v_1^-\psi_+^\pi \\ &+ G\psi_+^{\pi\dagger}\psi_+^\pi D_1\psi_+^{\pi\dagger}D_1\psi_+^\pi, \end{aligned} \quad (3.57)$$

where G is a four-fermion coupling constant. It should be noted that the effective coupling constants M , M' , N , and G are real. Since in the above Lagrangian, we have omitted the electron degrees of freedom, it is no longer $SU(2)_Q$ -invariant. Interestingly, the low-energy effective Lagrangian has an emergent Galilean boost symmetry, despite the fact that the underlying microscopic model does not possess this invariance.

I. Determination of the fermion mass parameters

We now like to match the fermion mass parameters to the parameters of the underlying microscopic system. Due to the $SU(2)_Q$ symmetry the masses of electrons and holes are identical. Here, we concentrate on the holes whose dispersion relation is given by

$$E_h(p) = M + \frac{p^2}{2M'}, \quad (3.58)$$

with the rest mass M and the kinetic mass M' . In the microscopic model in Eq. (2.20), we calculated the dispersion relations of holes

$$E_h(p) = \frac{J}{2} + \frac{U}{2} - 2t + ta^2(p - \pi/a)^2 + \mathcal{O}(((p - \pi/a)^4)). \quad (3.59)$$

One should keep in mind that the holes live in momentum space pockets centered at $p = \pi/a$, which must be taken into account in the matching of the parameters. Indeed, we had already identified the rest and kinetic masses as

$$M = \frac{J}{2} + \frac{U}{2} - 2t, \quad M' = \frac{1}{2ta^2}. \quad (3.60)$$

IV. NONPERTURBATIVE SOLUTION OF THE EFFECTIVE THEORY IN THE TWO-HOLE SECTOR

In this section, we will investigate the two-hole sector in the effective field theory and will then again compare with the analytic results of the underlying microscopic model.

Solution of the two-hole Schrödinger equation

In order to calculate the bound- and scattering-states of two holes in the effective theory, one can derive a two-hole potential from the effective Lagrangian of Eq. (3.57). The four-fermion contact term of strength G gives rise to a potential that is proportional to the second derivative of a δ function. Such potentials are ultraviolet divergent and require renormalization even in quantum mechanics. In order to avoid the corresponding subtleties, it is more efficient to apply the technique of self-adjoint extensions. In particular, it is then not even necessary to explicitly construct the potential. It should be mentioned that the method of self-adjoint extensions would be applicable also in higher dimensions.

Since the effective theory has an emergent Galilean boost symmetry, we may consider the two-hole system in its rest frame. Introducing the relative coordinate x between the two holes, the Schrödinger equation reduces to a single particle equation with the reduced mass $M'/2$. For kinematical reasons, the two holes cannot exchange magnons. Instead, they just experience their four-fermion contact interaction. Away from the contact point $x=0$, the two-hole Schrödinger equation thus describes free particles and is simply given by

$$-\frac{1}{M'}\partial_x^2\psi(x) = E\psi(x). \quad (4.1)$$

In the theory of self-adjoint extensions, contact interactions in one-dimensional quantum mechanics are treated by removing the contact point $x=0$ from the physical space. The effect of the four-fermion interaction is then represented by a boundary condition on the wave function. The most general self-adjoint extension has four independent parameters and is characterized by the boundary condition

$$\begin{pmatrix} \psi(\epsilon) \\ \partial_x \psi(\epsilon) \end{pmatrix} = \exp(i\theta) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi(-\epsilon) \\ \partial_x \psi(-\epsilon) \end{pmatrix}. \quad (4.2)$$

Here, a , b , c , and d are real numbers with the constraint $ad-bc=1$, and ϵ is an infinitesimal displacement from the point $x=0$. Since our system is parity-invariant (against the reflection R), the situation simplifies further and one obtains

$$\begin{pmatrix} \psi(\epsilon) \\ \partial_x \psi(\epsilon) \end{pmatrix} = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \begin{pmatrix} \psi(-\epsilon) \\ \partial_x \psi(-\epsilon) \end{pmatrix}, \quad (4.3)$$

i.e., $\theta=0$ and $d=a$. Since we are dealing with fermions, the Pauli principle implies a parity-odd wave function obeying $\psi(-x)=-\psi(x)$. The boundary condition on the wave function then implies

$$\kappa\psi(\epsilon) + \partial_x \psi(\epsilon) = 0, \quad \kappa = \frac{a+1}{b}. \quad (4.4)$$

It should be pointed out that the wave function will in general not be continuous at $x=0$. Alternatively to the four-fermion coupling G , the strength of the two-hole contact interaction can be characterized by the parameter κ . Relating κ to G would require the ultraviolet regularization of the second derivative of a δ -function potential. We avoid this unnecessary step by matching the value of κ directly to the parameters of the underlying microscopic model.

Let us first search for two-hole bound states (with $E < 0$). The wave function then takes the form

$$\psi(x) = A \exp(-\kappa x), \quad x > 0, \quad \psi(-x) = -\psi(x), \quad (4.5)$$

which is indeed discontinuous at $x=0$. The corresponding wave function in the microscopic model was calculated in Eq. (2.64) and (in the rest frame, i.e., for $p=0$) is given by

$$g(x) = C_0 (-1)^{x/a} \left(\frac{8t}{J} \right)^{x/a}. \quad (4.6)$$

The oscillating factor $(-1)^{x/a}$ is not present in the effective field theory because of the momentum shift in the effective hole fields which are located near $p=\pi/a$ in the Brillouin zone. Matching the exponential decays, we identify

$$\kappa = -\frac{1}{a} \log \left(\frac{8t}{J} \right). \quad (4.7)$$

In the effective theory, the bound-state energy is given by

$$E_B = -E = \frac{\kappa^2}{M'}. \quad (4.8)$$

The corresponding expression in the microscopic model was calculated in Eq. (2.63) and is given by

$$E_B = \frac{J}{4} \left(1 - \frac{8t}{J} \right)^2 = \frac{J}{8ta^2 M'_h} \left(1 - \frac{8t}{J} \right)^2. \quad (4.9)$$

Here, we have used the value $M'_h = 1/2ta^2$ for the kinetic hole mass. Hence, from this expression, one would conclude that

$$\kappa = \frac{1}{a} \sqrt{\frac{J}{8t}} \left(1 - \frac{8t}{J} \right). \quad (4.10)$$

This is consistent with Eq. (4.7) only when $J \approx 8t$, i.e., when the binding energy of Eq. (4.9) is small. In fact, the two expressions even coincide up to second order in the perturbation $\delta = 1 - 8t/J$. The effective field theory thus provides a correct description of the bound state only when the binding is weak. This is not surprising. If two holes form a bound state with a large binding energy, this bound state must be introduced in the effective theory as an independent degree of freedom. Only when the bound state resembles a weakly coupled ‘‘molecule’’ in which the constituent holes can be identified as relevant low-energy degrees of freedom, the effective field theory (without explicit bound-state fields) is appropriate. In this context, it is interesting to note that the kinetic mass of two holes calculated in Eq. (2.62) was given by

$$M'_{hh} = \frac{J}{8t^2 a^2}. \quad (4.11)$$

Only for $J=8t$, this corresponds to the sum of the kinetic masses of two holes $2M'_h = 1/ta^2$. This is consistent, because the emergent Galilean boost invariance of the effective theory indeed implies this relation.

Finally, let us consider the scattering states of two holes (with $E > 0$). We make the ansatz

$$\psi(x) = A \exp(ikx) + B \exp(-ikx), \quad (4.12)$$

insert it into Eq. (4.4), and find

$$\frac{A}{B} = \frac{k+i\kappa}{k-i\kappa}. \quad (4.13)$$

Let us now compare this result with the one obtained in the microscopic model given in Eq. (2.69). For low energies, i.e., for $q \rightarrow \pi/a$, one obtains

$$\frac{\tilde{A}}{\tilde{B}} = \frac{k+i(1-8t/J)}{k-i(1-8t/J)} + \mathcal{O}(k^2). \quad (4.14)$$

Indeed, using the value of κ given in Eq. (4.7), we finally get

$$\frac{A}{B} = \frac{\tilde{A}}{\tilde{B}}. \quad (4.15)$$

We conclude that also here, the effective field theory makes correct predictions, provided that the energies of both the bound state and the scattering states are small.

V. CONCLUSIONS

We have investigated a Hubbard-type model for a doped ferromagnet. While this model does not provide a realistic description of actual ferromagnetic systems, since it can be solved completely analytically, it provides a stringent test of the corresponding low-energy effective field theory for magnons and doped electrons or holes. Similar effective theories have been constructed for magnons and charge carriers in the

antiferromagnetic precursors of high-temperature superconductors. Since, in that case, the underlying microscopic models cannot be solved analytically, the correctness of the effective field theory can only be tested in Monte Carlo simulations. Indeed, such tests provide excellent numerical evidence for the validity of the effective field theory approach. In the ferromagnetic case discussed here, the exact agreement between the analytic results of the microscopic and the effective theory lends further support to the validity of the systematic low-energy effective field theory technique. In particular, we like to stress once more that the basic principles behind the construction of the effective theory are the same for ferro- and for antiferromagnets.

While in this work we have investigated bound and scattering states of two holes, another case of interest concerns the interaction between a spin wave and a hole. Indeed, in the microscopic theory one is then lead to a Faddeev-type equation and it would be instructive to confront the microscopic result with the effective theory prediction also for this case.

Effective field theories are also being used in the description of light nuclei. In that case, a low-energy effective field theory of pions and nucleons must be solved nonperturbatively, and it is currently not completely clear how to do this in a fully systematic manner, i.e., based on a consistent power-counting scheme. In this context, it is interesting that the two-hole sector of the ferromagnetic model discussed here can be solved nonperturbatively both in the microscopic and in the effective field theory treatment. Both approaches agree as long as the two-hole binding energy is small. On the other hand, when the binding becomes strong, the bound state should be described by an independent effective field. The analytically solvable test case of the ferromagnet may also provide valuable insights into the subtle power-counting issues that arise in the context of the strong interactions.

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APPENDIX: INTEGRALS FOR THE GAP EQUATION

In order to solve the gap Eqs. (2.35), we needed the integrals

$$I_1 = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq \frac{1}{\alpha \cos(qa) + \beta \sin(qa) + \gamma}$$

$$= \text{sign}(\gamma) \frac{1}{s},$$

$$I_2 = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq \frac{\cos(qa)}{\alpha \cos(qa) + \beta \sin(qa) + \gamma}$$

$$= -\frac{\alpha}{s(|\gamma| + s)},$$

$$I_3 = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq \frac{\sin(qa)}{\alpha \cos(qa) + \beta \sin(qa) + \gamma}$$

$$= -\frac{\beta}{s(|\gamma| + s)},$$

$$I_4 = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq \frac{\cos^2(qa)}{\alpha \cos(qa) + \beta \sin(qa) + \gamma}$$

$$= \text{sign}(\gamma) \frac{\alpha^2 + s(|\gamma| + s)}{s(|\gamma| + s)^2},$$

$$I_5 = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq \frac{\sin^2(qa)}{\alpha \cos(qa) + \beta \sin(qa) + \gamma}$$

$$= \text{sign}(\gamma) \frac{\beta^2 + s(|\gamma| + s)}{s(|\gamma| + s)^2},$$

$$I_6 = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq \frac{\cos(qa)\sin(qa)}{\alpha \cos(qa) + \beta \sin(qa) + \gamma}$$

$$= \text{sign}(\gamma) \frac{\alpha\beta}{s(|\gamma| + s)^2},$$

$$s = \sqrt{\gamma^2 - \alpha^2 - \beta^2}. \quad (\text{A1})$$

The aim of this appendix is to show how these integrals can be evaluated. Since

$$\alpha I_2 + \beta I_3 + \gamma I_1 = 1, \quad I_4 + I_5 = I_1, \quad (\text{A2})$$

one only needs to do four integrals, for example, I_1 , I_2 , I_4 , and I_6 . These four integrals can be evaluated by using the residue theorem. We now explicitly present the calculation for

$$I_1 = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dq [\alpha \cos(qa) + \beta \sin(qa) + \gamma]^{-1}. \quad (\text{A3})$$

We integrate around the unit circle C , and thus make the substitution

$$z = \exp(ika).$$

Then, we can write

$$I_1 = \frac{1}{2\pi i} \oint_C dz \left[\frac{\alpha}{2}(z^2 + 1) + \frac{\beta}{2i}(z^2 - 1) + \gamma z \right]^{-1}. \quad (\text{A4})$$

The integrand has two singularities at

$$z_A = \frac{-\gamma + \sqrt{\gamma^2 - \alpha^2 - \beta^2}}{\alpha - \beta i},$$

$$z_B = \frac{-\gamma - \sqrt{\gamma^2 - \alpha^2 - \beta^2}}{\alpha - \beta i}. \quad (\text{A5})$$

We get a contribution to the integral only if the singularities lie within the unit circle C . Therefore, we consider the absolute values of z_A and z_B ,

$$|z_A|^2 = \frac{(\sqrt{\gamma^2 - \alpha^2 - \beta^2} - \gamma)^2}{\alpha^2 + \beta^2},$$

$$|z_B|^2 = \frac{(\sqrt{\gamma^2 - \alpha^2 - \beta^2} + \gamma)^2}{\alpha^2 + \beta^2}. \quad (\text{A6})$$

We find

$$|z_A|^2 < 1 \Leftrightarrow \gamma > 0, \quad |z_B|^2 > 1 \Leftrightarrow \gamma > 0, \quad (\text{A7})$$

as well as

$$|z_A|^2 > 1 \Leftrightarrow \gamma < 0, \quad |z_B|^2 < 1 \Leftrightarrow \gamma < 0. \quad (\text{A8})$$

Hence, for $\gamma > 0$ only z_A lies within the unit circle C and for $\gamma < 0$ only z_B lies within C . The residues of the poles at z_A and z_B are given by

$$R_A = \frac{-2\gamma + 2\sqrt{\gamma^2 - \alpha^2 - \beta^2}}{2\alpha\sqrt{\gamma^2 - \alpha^2 - \beta^2} - 2i\beta\sqrt{\gamma^2 - \alpha^2 - \beta^2}},$$

$$R_B = \frac{2\gamma + 2\sqrt{\gamma^2 - \alpha^2 - \beta^2}}{2\alpha\sqrt{\gamma^2 - \alpha^2 - \beta^2} - 2i\beta\sqrt{\gamma^2 - \alpha^2 - \beta^2}}. \quad (\text{A9})$$

Collecting the results, we obtain

$$I_1 = \text{sign}(\gamma) \frac{1}{\sqrt{\gamma^2 - \alpha^2 - \beta^2}}. \quad (\text{A10})$$

The integrals I_2 , I_4 , and I_6 can be obtained completely analogously.

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